Mathematical Physics

JEST-2012

Q1. The value of the integral $\int_0^{\infty} \frac{\ln x}{\left(x^2+1\right)^2} dx$ is

(a) 0 (b) $\frac{-\pi}{4}$ (c) $\frac{-\pi}{2}$ (d) $\frac{\pi}{2}$

Ans. : (b)

Solution: $\int\limits_0^{\infty} \frac{\ln x}{\left(x^2+1\right)^2} dx = \int\limits_0^{\infty} \frac{\ln z}{\left(z^2+1\right)^2} dz$

Let us consider new function $f(z)=\left(\frac{\ln z}{z^2+1}\right)^2$, then $I=\int\limits_0^{\infty}\left(\frac{\ln z}{z^2+1}\right)^2 dz$

Pole at $z=\pm i$ is simple pole of second order.

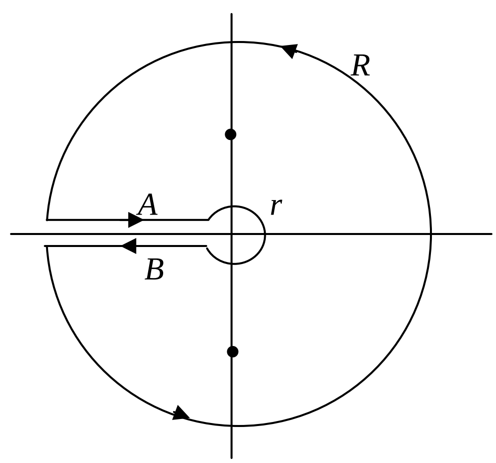
Residue at $z=i$ is

$$=\frac{d}{dz}(z-i)^2 \frac{(\ln z)^2}{(z-i)^2(z+i)^2}=\frac{d}{dz}\frac{(\ln z)^2}{(z+i)^2}$$

$$=\frac{(z+i)^2 2(\ln z).\frac{1}{z}-(\ln z)^2 .2(z+i)}{(z+i)^4}=\frac{(z+i)2\ln(z)\frac{1}{z}-(\ln z)^2 .2}{(z+i)^3}$$

$$=\frac{(2i)2\times\frac{1}{i}\ln i-(\ln i)^2\cdot 2}{(2i)^3}=\frac{4\frac{i\pi}{2}-\left(\frac{i\pi}{2}\right)^2\times 2}{-8i}=\frac{2\pi i+\frac{\pi^2}{2}}{-8i}$$

$$\Rightarrow \text{Res}\big|_{z=i}=\frac{-\pi}{4}+\frac{\pi^2}{16}i$$



Similarly, at $z=-i$; $\text{Res}\big|_{z=-i}=\frac{-\pi}{4}-\frac{\pi^2}{16}i$

$$I=\int\limits_0^{\infty}\left(\frac{\ln z}{z^2+1}\right)^2 dz=2\pi i\left(\frac{-\pi}{4}+\frac{\pi^2}{16}i-\frac{\pi}{4}-\frac{\pi^2}{16}i\right)=-\pi^2 i$$

$$-\pi^2 i=\left(\int_R\int_A\int_B\int_r\right) f(z)\,dz=\left(\int_A\int_B\right) f(z)\,dz; \qquad \left[\because \int_A\int_B \text{ vanish}\right]$$

Along path A; $z=-x+i\varepsilon$ and along path B; $z=-x-i\varepsilon$

$$\text{Thus } -\pi^2 i=\left(\int_A\int_B\right)f(z)\,dz=-\int_{-\infty}^{0}\left[\frac{\ln(-x+i\varepsilon)}{(-x+i\varepsilon)^2+1}\right]dx-\int_0^{\infty}\left[\frac{\ln(-x-i\varepsilon)}{(-x-i\varepsilon)^2+1}\right]dx$$

$$\Rightarrow -\pi^2 i=\int_0^{\infty}\left[\frac{\ln(-x+i\varepsilon)}{(-x+i\varepsilon)^2+1}\right]^2dx-\int_0^{\infty}\left[\frac{\ln(-x-i\varepsilon)}{(-x-i\varepsilon)^2+1}\right]^2dx$$

$$\Rightarrow -\pi^2 i=\int_0^{\infty}\left[\frac{\ln(x)+i\pi}{1+x^2}\right]^2dx-\int_0^{\infty}\left[\frac{\ln(x)-i\pi}{1+x^2}\right]^2dx;\quad \varepsilon\to 0$$

$$\Rightarrow -\pi^2 i=\int_0^{\infty}\frac{(\ln(x)+i\pi)^2-(\ln(x)-i\pi)^2}{(1+x^2)^2}dx=4\pi i\int_0^{\infty}\frac{\ln x}{(x^2+1)^2}\Rightarrow\int_0^{\infty}\frac{\ln x}{(x^2+1)^2}=\frac{-i\pi^2}{4\pi i}=\frac{-\pi}{4}$$

Q2. If [x] denotes the greatest integer not exceeding x, then $\int_0^{\infty}[x]e^{-x}dx$

(a) $\frac{1}{e-1}$ (b) 1 (c) $\frac{e-1}{e}$ (d) $\frac{e}{e^2-1}$

Ans.: (a)

Solution: $[x]$

$0\le x<1=[x]=0$, $1\le x<2=[x]=1$, $2\le x<3=[x]=2$

$$\text{Now, }\int_0^{\infty}[x]e^{-x}dx=\int_0^{1}[x]e^{-x}dx+\int_1^{2}[x]e^{-x}dx+\int_2^{3}[x]e^{-x}dx+\int_3^{4}[x]e^{-x}dx$$

$$\Rightarrow 0+\int_1^2 1.e^{-x}dx+\int_2^3 2.e^{-x}dx+\int_3^4 3.e^{-x}dx=\left[-e^{-x}\right]_1^2+2\left(-e^{-x}\right)_2^3+3\left(-e^{-x}\right)_3^4+\ldots.$$

$$=e^{-1}-e^{-2}+2e^{-2}-2e^{-3}+3e^{-3}-3e^{-4}+4e^{-4}-4e^{-5}+$$

$$=e^{-1}+e^{-2}+e^{-3}+e^{-4}+\ldots..\infty$$

$$=\frac{e^{-1}}{1-e^{-1}}=\frac{1}{e-1}\qquad\left(\because\ r=\frac{e^{-2}}{e^{-1}}=e^{-2+1}=e^{-1}\right)$$

Q3. As $x \to 1$, the infinite series $x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + ...$

(a) diverges (b) converges to unity

(c) converges to $\frac{\pi}{4}$ (d) none of the above

Ans.: (c)

Solution: $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... \Rightarrow \tan^{-1} 1 = \frac{\pi}{4}$

Q4. What is the value of the following series?

$$\left(1+\frac{1}{2!}+\frac{1}{4!}+....\right)^2 - \left(1+\frac{1}{3!}+\frac{1}{5!}+.....\right)^2$$

(a) 0 (b) e (c) e^2 (d) 1

Ans.: (d)

Solution: $e^1 = 1+1+\frac{1^2}{2!}+\frac{1^3}{3!}+-$, $e^{-1} = 1-1+\frac{1^2}{2!}-\frac{1^3}{3!}.....$

$$\cosh 1 = \frac{e^1 + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + ...$$

$$\sinh 1 = \frac{\left(e^1 - e^{-1}\right)}{2} = 1 + \frac{1}{3!} + \frac{1}{5!} + ...$$

i.e., $\cosh^2 1 - \sinh^2 1 = 1$

Q5. An unbiased die is cast twice. The probability that the positive difference (bigger - smaller) between the two numbers is 2 is

(a) $\frac{1}{9}$ (b) $\frac{2}{9}$ (c) $\frac{1}{6}$ (d) $\frac{1}{3}$

Ans.: (b)

Solution: $p(2) = \frac{n(E)}{n(S)}$

The number of ways to come positive difference

$$\left[(3,\ 1),\ (4,\ 2),\ (5,\ 3), (6,\ 4), (1,3)(2,4), (3,5)(4,6)\right]$$

$$p(2) = \frac{8}{36} = \frac{2}{9}$$

Q6. For an $N \times N$ matrix consisting of all ones,

(a) all eigenvalues $=1$ (b) all eigenvalues $=0$

(c) the eigenvalues are $1, 2, \ldots . N$ (d) one eigenvalue $=N$, the others $=0$

Ans.: (d)

Solution: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0, \; 2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 0, 0, 3$$

So, far $N \times N$ matrix one eigen value is N and all other eigen values are zero.

JEST-2013

Q7. A box contains 100 coins out of which 99 are fair coins and 1 is a double-headed coin. Suppose you choose a coin at random and toss it 3 times. It turns out that the results of all 3 tosses are heads. What is the probability that the coin you have drawn is the double-headed one?

(a) 0.99 (b) 0.925 (c) 0.75 (d) 0.01

Ans.: (c)

Q8. Compute $\lim\limits_{z \to 0} \dfrac{\operatorname{Re}\left(z^2\right) + \operatorname{Im}\left(z^2\right)}{z^2}$

(a) The limit does not exist (b) 1

(c) $-i$ (d) -1

Ans.: (a)

Solution: $\lim\limits_{z \to 0} \dfrac{\operatorname{Re}\left(z^2\right) + \operatorname{Im}\left(z^2\right)}{z^2} = \lim\limits_{z \to 0} \dfrac{x^2 - y^2 + 2xy}{x^2 - y^2 + 2ixy} = \lim\limits_{\substack{y=0 \\ x \to 0}} \dfrac{x^2 - y^2 + 2xy}{x^2 - y^2 + 2ixy} = 1$

$$\lim_{\substack{x=0 \\ y \to 0}} \frac{x^2 - y^2 + 2xy}{x^2 - y^2 + 2ixy} = 1 \quad \text{and} \quad \lim_{\substack{y=x \\ x \to 0}} \frac{x^2 - y^2 + 2xy}{x^2 - y^2 + 2ixy} = -i$$

Q9. The vector field $xz\hat{i}+y\hat{j}$ in cylindrical polar coordinates is

(a) $\rho\left(z\cos^2\phi+\sin^2\phi\right)\hat{e}_\rho+\rho\sin\phi\cos\phi(1-z)\hat{e}_\phi$

(b) $\rho\left(z\cos^2\phi+\sin^2\phi\right)\hat{e}_\rho+\rho\sin\phi\cos\phi(1+z)\hat{e}_\phi$

(c) $\rho\left(z\sin^2\phi+\cos^2\phi\right)\hat{e}_\rho+\rho\sin\phi\cos\phi(1+z)\hat{e}_\phi$

(d) $\rho\left(z\sin^2\phi+\cos^2\phi\right)\hat{e}_\rho+\rho\sin\phi\cos\phi(1-z)\hat{e}_\phi$

Ans.: (a)

Solution: $\vec{A}=xz\hat{i}+y\hat{j}\Rightarrow A_x=xz,\quad A_y=y,\quad A_z=0$

$$A_\rho=\vec{A}\cdot\hat{e}_\rho=A_x\left(\hat{x}\cdot\hat{e}_\rho\right)+A_y\left(\hat{y}\cdot\hat{e}_\rho\right)+A_z\left(\hat{z}\cdot\hat{e}_\rho\right)$$

$$\Rightarrow A_\rho=\rho\cos\phi z\left(\cos\phi\right)+\rho\sin\phi\left(\sin\phi\right)+0\Rightarrow A_\rho=\left(\rho\cos\phi^2 z+\rho\sin^2\phi\right)\hat{e}_\rho$$

$$A_\phi=\vec{A}\cdot\hat{e}_\phi=A_x\left(\hat{x}\cdot\hat{e}_\phi\right)+A_y\left(\hat{y}\cdot\hat{e}_\phi\right)+A_z\left(\hat{z}\cdot\hat{e}_\phi\right)$$

$$\Rightarrow A_\phi=\rho\cos\phi\left(-\sin\phi\right)z+\rho\sin\phi\cdot\cos\phi\Rightarrow A_\phi=\rho\cos\phi\cdot\sin\phi\left(1-z\right)\hat{e}_\phi$$

$$\vec{A}=A_\rho\hat{e}_\rho+A_\phi\hat{e}_\phi+A_z\hat{e}_z=\rho\left(z\cos^2\phi+\sin^2\phi\right)\hat{e}_\rho+\rho\cos\phi\sin\phi\left(1-z\right)\hat{e}_\phi$$

Q10. There are on average 20 buses per hour at a point, but at random times. The probability that there are no buses in five minutes is closest to

(a) 0.07 (b) 0.60 (c) 0.36 (d) 0.19

Ans.: (d)

Solution: From Poision's distribution function,

$$P(n)=\frac{e^{-\lambda}\lambda^n}{\underline{|n}}$$

here, $\lambda=20$ buses per hour

$\Rightarrow\lambda=\frac{5}{3}$ buses in five minutes

Therefore, the probability that there are no buses in five minutes,

$$P(n=0)=\frac{e^{-\frac{5}{3}}\left(\frac{5}{3}\right)^0}{\underline{|0}}=e^{-5/3}=0.1886\approx 0.19$$

Thus, option (d) is correct option.

Q11. Two drunks start out together at the origin, each having equal probability of making a step simultaneously to the left or right along the x axis. The probability that they meet after n steps is

(a) $\frac{1}{4^n}\frac{2n!}{n!^2}$ (b) $\frac{1}{2^n}\frac{2n!}{n!^2}$ (c) $\frac{1}{2^n}2n!$ (d) $\frac{1}{4^n}n!$

Ans.: (a)

Solution: The probability of taking $'r'$ steps out of N steps $= {}^{N}C_r\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{N-r}$

Total steps $= N = n + n = 2n$

For taking probability of n steps out of N

$$P = {}^{N}C_n\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^{N-n} = \frac{N!}{(N-n)!n!}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^{N-n} = \frac{2n!}{n!n!}\left(\frac{1}{2}\right)^{2n} = \frac{2n!}{(n!)^2 4^n}$$

Q12. What is the value of the following series?

$$\left(1-\frac{1}{2!}+\frac{1}{4!}-....\right)^2+\left(1-\frac{1}{3!}+\frac{1}{5!}-...\right)^2$$

(a) 0 (b) e (c) e^2 (d) 1

Ans.: (d)

Solution: $\cos\theta = 1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}.....$, $\sin\theta = \theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}.....$

$$\Rightarrow\left(1-\frac{1}{2!}+\frac{1}{4!}...\right)^2+\left(1-\frac{1}{3!}+\frac{1}{5!}\right)^2 = \cos^2 1+\sin^2 1 = 1 \qquad \left[\because \sin^2\theta+\cos^2\theta = 1\right]$$

Q13. If the distribution function of x is $f(x) = xe^{-x/\lambda}$ over the interval $0 < x < \infty$, the mean value of x is

(a) λ (b) 2λ (c) $\frac{\lambda}{2}$ (d) 0

Ans.: (b)

Solution: Since, it is distribution function so,

$$\langle x\rangle = \frac{\int_{-\infty}^{\infty} xf(x)dx}{\int_{-\infty}^{\infty} f(x)dx} = \frac{\int_0^{\infty} x.xe^{-\frac{x}{\lambda}}dx}{\int_0^{\infty} xe^{-\frac{x}{\lambda}}dx} = \frac{\int_0^{\infty} x^2e^{\frac{-x}{\lambda}}dx}{\int_0^{\infty} xe^{\frac{-x}{\lambda}}dx} = 2\lambda$$

JEST-2014

Q14. What are the solutions of $f''(x)-2f'(x)+f(x)=0$?

(a) c_1e^x/x (b) c_1x+c_2/x (c) $c_1xe^x+c_2$ (d) $c_1e^x+c_2xe^x$

Ans.: (d)

Solution: Auxiliary equation is, $D^2-2D+1=0\Rightarrow(D-1)^2=0\Rightarrow D=+1,\ +1$

$\because$ Roots are equal, then $f(x)=(c_1+c_2x)e^x\Rightarrow f(x)=c_1e^x+c_2xe^x$

Q15. The value of $\int_{0.2}^{2.2}xe^x dx$ by using the one-segment trapezoidal rule is closed to

(a) 11.672 (b) 11.807 (c) 20.099 (d) 24.119

Ans.: (c)

Solution: $h=2.2-0.2=2\Rightarrow I=\frac{h}{2}\left[y(2.2)+y(0.2)\right]=20.099$ $\left[\because y=xe^x\right]$

Q16. Given the fundamental constants $\hbar$ (Planck's constant), G (universal gravitation constant) and c (speed of light), which of the following has dimension of length?

(a) $\sqrt{\frac{\hbar G}{c^3}}$ (b) $\sqrt{\frac{\hbar G}{c^5}}$ (c) $\frac{\hbar G}{c^3}$ (d) $\sqrt{\frac{\hbar c}{8\pi G}}$

Ans.: (a)

Solution: $\left[\frac{\left[ML^2T^{-1}\right]\left[M^{-1}L^3T^{-2}\right]}{L^3T^{-3}}\right]^{\frac{1}{2}}=\left[L^2\right]^{\frac{1}{2}}=L$

$\hbar=\left[ML^2T^{-1}\right],\ G=\frac{gr^2}{m}=\left[M^{-1}L^3T^{-2}\right]$

Q17. The Laplace transformation of $e^{-2t}\sin 4t$ is

(a) $\frac{4}{s^2+4s+25}$ (b) $\frac{4}{s^2+4s+20}$

(c) $\frac{4s}{s^2+4s+20}$ (d) $\frac{4s}{2s^2+4s+20}$

Ans.: (b)

Solution: $\because L\left[e^{-at}\sin bt\right]=\frac{b}{(s+a)^2+b^2}$

$\Rightarrow L\left[e^{-2t}\sin 4t\right]=\frac{4}{(s+2)^2+4^2}=\frac{4}{s^2+4s+20}$

Q18. Let us write down the Lagrangian of a system as $L(x,\ddot{x},\dddot{x}) = m x\ddot{x} + kx^2 + c x\dddot{x}$. What is the dimension of c?

(a) MLT^{-3} (b) MT^{-2} (c) MT (d) ML^2T^{-1}

Ans.: (c)

Solution: According to dimension rule same dimension will be added or subtracted then dimension of $Mx\ddot{x} =$ dimension of $Cx\dddot{x}$

$$\left[ML^2T^{-2}\right] = [C][L]\left[LT^{-3}\right]$$

$$[C] = \frac{\left[ML^2T^{-2}\right]}{\left[L^2T^{-3}\right]} = [MT]$$

Q19. The Dirac delta function $\delta(x)$ satisfies the relation $\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$ for a well behaved function $f(x)$. If x has the dimension of momentum then

(a) $\delta(x)$ has the dimension of momentum

(b) $\delta(x)$ has the dimension of $(\text{momentum})^2$

(c) $\delta(x)$ is dimensionless

(d) $\delta(x)$ has the dimension of $(\text{momentum})^{-1}$

Ans.: (d)

Solution: $\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$

$$f(x)\,\delta(x)\,dx = f(0) \Rightarrow [f(x)]\delta(x)\cdot P = [f(0)] \Rightarrow \delta(x) = \left[P^{-1}\right]$$

Since, $[f(x)] = [f(0)]$

If $F(x) = \alpha x + \beta$ is force $\left[M\,L\,T^{-2}\right]$

$F(0) = \beta$ is also $\left[M\,L\,T^{-2}\right]$

Q20. The value of limit

$$\lim_{z\to i}\frac{z^{10}+1}{z^6+1}$$

is equal to

(a) 1 (b) 0 (c) $\frac{-10}{3}$ (d) $\frac{5}{3}$

Ans.: (d)

Solution: $\lim_{z\to i}\frac{z^{10}+1}{z^6+1} = \lim_{z\to i}\frac{10z^9}{6z^5} = \lim_{z\to i}\frac{10z^4}{6} = \frac{10}{6} = \frac{5}{3}$

Q21. The value of integral

$$I = \oint_c \frac{\sin z}{2z - \pi} dz$$

with c a circle $|z| = 2$, is

(a) 0 (b) $2\pi i$ (c) πi (d) $-\pi i$

Ans.: (c)

Solution: $I = \oint_C \frac{\sin z}{2z - \pi}$, for pole $2z - \pi = 0 \Rightarrow z = \frac{\pi}{2}$

Residue at $z = \frac{\pi}{2}$ $\because |z| = 2$, so pole will lie within the contour

$$I = \oint_C \frac{e^{iz}}{2\left(z - \frac{\pi}{2}\right)} = \sum R \times 2\pi i$$

$$\text{Res}\left|_{z=\frac{\pi}{2}} = \frac{\left(z - \frac{\pi}{2}\right) e^{iz}}{2\left(z - \frac{\pi}{2}\right)} = \frac{e^{i\pi/2}}{2} = \frac{i}{2}\right.$$ (taking imaginary part); Residue = $\frac{1}{2}$

Now, $I = \frac{1}{2} \times 2\pi i = \pi i$

JEST-2015

Q22. Given an analytic function $f(z) = \phi(x, y) + i\psi(x, y)$, where $\phi(x, y) = x^2 + 4x - y^2 + 2y$.

If C is a constant, which of the following relations is true?

(a) $\psi(x, y) = x^2 y + 4y + C$ (b) $\psi(x, y) = 2xy - 2x + C$

(c) $\psi(x, y) = 2xy + 4y - 2x + C$ (d) $\psi(x, y) = x^2 y - 2x + C$

Ans. : (c)

Solution: $u = \phi(x, y) = x^2 + 4x - y^2 + 2y$, $v = \psi$

From C.R. equation, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\Rightarrow \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$

Now, $\frac{\partial \phi}{\partial x} = 2x + 4 = \frac{\partial \psi}{\partial y}$

$\Rightarrow \psi = 2xy + 4y + f(x)$ (i)

and $\frac{\partial \phi}{\partial y} = -2y + 2 \Rightarrow \frac{\partial \psi}{\partial x} = +2y - 2$

$\psi = 2xy + 2x + f(y)$ (ii)

From (i) and (ii), $2xy + 4y + f(x) = 2xy - 2x + f(y)$

$f(x) = -2x, \quad f(y) = 4y$

$\psi = 2xy + 4y - 2x + c$

Q23. If two ideal dice are rolled once, what is the probability of getting atleast one '6'?

(a) $\frac{11}{36}$ (b) $\frac{1}{36}$ (c) $\frac{10}{36}$ (d) $\frac{5}{36}$

Ans: (a)

Solution: Number of point in sample space $n(S) = 11$

$\left[(1,6),(2,6),(3,6),(4,6),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\right]$

Number of point in population $n(P) = 6^2 = 36$

Probability of getting atleast one '6' on face of dice $= \frac{n(S)}{n(P)} = \frac{11}{36}$

Q24. What is the maximum number of extrema of the function $f(x) = P_k(x)e^{-\left(\frac{x^4}{4}+\frac{x^2}{2}\right)}$, where $x \in (-\infty, \infty)$ and $P_k(x)$ is an arbitrary polynomial of degree k?

(a) $k+2$ (b) $k+6$ (c) $k+3$ (d) k

Ans.: (c)

Solution: $f(x) = P_x(x)e^{-\left(\frac{x^4}{4}+\frac{x^2}{2}\right)}$

$f'(x) = \left[P_x'(x) + P_x(x)(-1)(x^3 + x)\right]e^{-\left(\frac{x^4}{4}+\frac{x^2}{2}\right)}$

For maximum number of extrema,

$\Rightarrow f'(x) = 0 \Rightarrow \left[P_x(x)(x^3 + x) - P'(x)\right] = 0$ is polynomial of order $k+3$

From the sign scheme maximum number of extrema $= k+3$

Q25. The Bernoulli polynominals $B_n(s)$ are defined by, $\frac{xe^{xs}}{e^x-1}=\sum B_n(s)\frac{x^n}{n!}$. Which one of the following relations is true?

(a) $\frac{xe^{x(1-s)}}{e^x-1}=\sum B_n(s)\frac{x^n}{(n+1)!}$

(b) $\frac{xe^{x(1-s)}}{e^x-1}=\sum B_n(s)(-1)^n\frac{x^n}{(n+1)!}$

(c) $\frac{xe^{x(1-s)}}{e^x-1}=\sum B_n(-s)(-1)^n\frac{x^n}{n!}$

(d) $\frac{xe^{x(1-s)}}{e^x-1}=\sum B_n(s)(-1)^n\frac{x^n}{n!}$

Ans.: (d)

Solution: $\frac{xe^{xS}}{e^x-1}=\sum B_n(s)\frac{x^n}{\lfloor n}$

Put $s=(s-1)$, $\frac{xe^{x(s-1)}}{e^x-1}=\sum B_n(s-1)\frac{x^n}{\lfloor n}$

Since, $B_n(s-1)=(-1)^n B(s)$

$\Rightarrow \frac{xe^{x(s-1)}}{e^x-1}=\sum B_n(s)(-1)^n\frac{x^n}{\lfloor n}$

Q26. Consider the differential equation $G'(x)+kG(x)=\delta(x)$; where k is a constant. Which of the following statements is true?

(a) Both $G(x)$ and $G'(x)$ are continuous at $x=0$.

(b) $G(x)$ is continuous at $x=0$ but $G'(x)$ is not.

(c) $G(x)$ is discontinuous at $x=0$.

(d) The continuity properties of $G(x)$ and $G'(x)$ at $x=0$ depends on the value of k.

Ans.: (c)

Q27. The sum $\sum_{m=1}^{99}\frac{1}{\sqrt{m+1}+\sqrt{m}}$ is equal to

(a) 9 (b) $\sqrt{99}-1$ (c) $\frac{1}{(\sqrt{99}-1)}$ (d) 11

Ans.: (a)

Solution: $\sum_{m=1}^{99}\frac{1}{\sqrt{m+1}+\sqrt{m}}=\sum_{m=1}^{99}\frac{\sqrt{m+1}-\sqrt{m}}{(m+1)-m}=\sum_{m=1}^{99}\sqrt{m+1}-\sqrt{m}$

$=\sqrt{2}-\sqrt{1}+\sqrt{3}-\sqrt{2}......+\sqrt{100}-\sqrt{99}=\sqrt{100}-\sqrt{1}=10-1=9$

JEST-2016

Q28. Given the condition $\nabla^2\phi = 0$, the solution of the equation $\nabla^2\psi = k\vec{\nabla}\phi.\vec{\nabla}\phi$ is given by

(a) $\psi = \frac{k\phi^2}{2}$ (b) $\psi = k\phi^2$ (c) $\psi = \frac{k\phi\ln\phi}{2}$ (d) $\psi = \frac{k\phi\ln\phi}{2}$

Ans.: (a)

Solution: $\nabla^2\phi = 0 \Rightarrow \vec{\nabla}.\left(\vec{\nabla}\phi\right) = 0 \Rightarrow \vec{\nabla}.\left(\vec{\nabla}\phi\right) = 0 \Rightarrow \vec{\nabla}\phi = \alpha\hat{x} + \beta\hat{y} + \gamma\hat{z} \Rightarrow \phi = \alpha x + \beta y + \gamma z$

$$k\vec{\nabla}\phi.\vec{\nabla}\phi = k\left(\alpha^2 + \beta^2 + \gamma^2\right)$$

If $\psi = \frac{k\phi^2}{2} = \frac{k}{2}(\alpha x + \beta y + \gamma z)^2$

$$\Rightarrow \nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = k\left(\alpha^2 + \beta^2 + \gamma^2\right) \Rightarrow \nabla^2\psi = k\vec{\nabla}\phi.\vec{\nabla}\phi$$

Q29. The mean value of random variable x with probability density $p(x) = \frac{1}{\sigma\sqrt{2\pi}}.\exp\left[-\frac{\left(x^2 + \mu x\right)}{\left(2\sigma^2\right)}\right]$ is:

(a) 0 (b) $\frac{\mu}{2}$ (c) $\frac{-\mu}{2}$ (d) σ

Ans. : (a)

Solution: $\langle x\rangle = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} x\exp\left(-\frac{x^2}{2\sigma^2}\right)dx\int_{-\infty}^{\infty}\exp\left(-\frac{\mu x}{2\sigma^2}\right)dx = 0$ (due to odd function)

Q30. Given a matrix $M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, which of the following represents $\cos\left(\frac{\pi M}{6}\right)$

(a) $\frac{1}{2}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ (b) $\frac{\sqrt{3}}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ (c) $\frac{\sqrt{3}}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (d) $\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$

Ans. : (b)

Solution: Given, $M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

For eigen value, $\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 1,3$

Now, for $\lambda = 1$,

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\Rightarrow 2x_1 + x_2 = x_1 \Rightarrow x_1 = -x_2$

Therefore, eigen vector associated with eigen value,

$$\lambda = 1 \text{ is } \phi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Similarly, for $\lambda = 3$, we get associated eigen vector as,

$$\phi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus, $M = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$\cos\frac{\pi}{6}M = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \cos\frac{\pi}{6} & 0 \\ 0 & \cos\frac{\pi}{2} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} \end{bmatrix}$$

Therefore, $\cos\left(\frac{\pi M}{6}\right) = \begin{bmatrix} \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} \end{bmatrix} = \frac{\sqrt{3}}{4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Thus, option (b) is correct option.

Q31. The sum of the infinite series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ is

(a) 2π (b) π (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

Ans. : (d)

Solution: The series expansion of $\tan^{-1} x$ in interval $-1 < x \le 1$ is,

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

Putting $x = 1$, we get,

$$\tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots = \frac{\pi}{4}$$

Thus, option (d) is correct.

Q32. A semicircular piece of paper is folded to make a cone with the centre of the semicircle as the apex. The half-angle of the resulting cone would be:

(a) 90° (b) 60° (c) 45° (d) 30°

Ans. : (d)

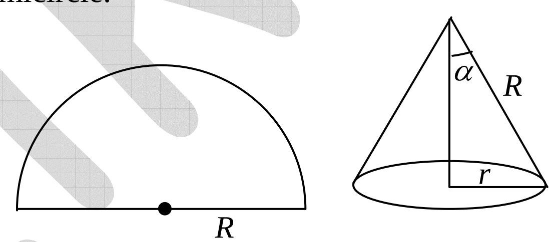
Solution: When the semicircular piece of paper is folded to make a cone, the circumference of base is equal to the circumference of the original semicircle. Let r be the radius of the base of the cone and R be the radius of the semicircle.

Hence, $2\pi r = \pi R \Rightarrow r = \frac{R}{2}$.

The stay height of the come will also be R.

Hence, $\sin\alpha = \frac{R/2}{R} = \frac{1}{2}$

Thus, $\alpha = 30^0$



JEST 2017

Q33. $\int_{-\infty}^{+\infty} (x^2 + 1)\delta(x^2 - 3x + 2)\,dx = ?$

(a) 1 (b) 2 (c) 5 (d) 7

Ans. : (d)

Solution: $\because\ (x^2 - 3x + 2) = (x^2 - x - 2x + 2) = x(x-1) - 2(x-1) = (x-1)(x-2)$

$$\Rightarrow \int_{-\infty}^{\infty} (x^2+1)\delta(x^2 - 3x + 2)\,dx = \int_{-\infty}^{\infty} (x^2+1)\delta[(x-1)(x-2)]\,dx$$

$$= \frac{1}{|2-1|}[f(1) + f(2)] = [2+5] = 7$$

Q34. Which one is the image of the complex domain $\{z|xy \geq 1, x+y>0\}$ under the mapping $f(z)=z^2$, if $z=x+iy$?

(a) $\{z|xy \geq 1, x+y>0\}$ (b) $\{z|x \geq 2, x+y>0\}$

(c) $\{z|y \geq 2\forall x\}$ (d) $\{z|y \geq 1\forall x\}$

Q35. Let $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 11 \end{pmatrix}$ and $M = \begin{pmatrix} 10 & 3i \\ -3i & 2 \end{pmatrix}$. Similarity, transformation of M to Λ can be performed by

(a) $\frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 3i \\ 3i & 1 \end{pmatrix}$ (b) $\frac{1}{\sqrt{9}}\begin{pmatrix} 1 & -3i \\ 3i & 11 \end{pmatrix}$

(c) $\frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 3i \\ -3i & 11 \end{pmatrix}$ (d) $\frac{1}{\sqrt{9}}\begin{pmatrix} 1 & 3i \\ -3i & 1 \end{pmatrix}$

Ans. : (a)

Solution: $M = \begin{pmatrix} 10 & 3i \\ -3i & 2 \end{pmatrix}$

The eigen value of matrix M is $1, 11$ and corresponding eigen vector are:

$|\phi_1\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 3i \end{pmatrix}, |\phi_2\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} 3i \\ 1 \end{pmatrix}$ respectively.

Now, $P = \left(|\phi_1\rangle \ |\phi_2\rangle \right)$

$$P = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 3i \\ 3i & 1 \end{pmatrix}$$

Q36. Suppose that we toss two fair coins hundred times each. The probability that the same number of heads occur for both coins at the end of the experiment is

(a) $\left(\frac{1}{4}\right)^{100} \sum_{n=0}^{100} \binom{100}{n}$ (b) $2\left(\frac{1}{4}\right)^{100} \sum_{n=0}^{100} \binom{100}{n}^2$

(c) $\frac{1}{2}\left(\frac{1}{4}\right)^{100} \sum_{n=0}^{100} \binom{100}{n}^2$ (d) $\left(\frac{1}{4}\right)^{100} \sum_{n=0}^{100} \binom{100}{n}^2$

Ans. : (d)

Solution: If we toss one fair coins hundred times, then probability of n number of head occurs at the end of 100 times is

$$^{100}C_n\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^{100-n}$$

Hence, the probability that same number of heads occur for both coins at the end of experiment is

$$\sum_{n=0}^{100}\left(^{100}C_n\left(\frac{1}{2}\right)^{100}\right).\left(^{100}C_n\left(\frac{1}{2}\right)^{100}\right)=\sum_{n=1}^{100}\left(^{100}C_n\right)^2\left(\frac{1}{2}\right)^{200}=\left(\frac{1}{4}\right)^{100}\sum_{n=1}^{100}\left(^{100}C_n\right)^2$$

Q37. What is the equation of the plane which is tangent to the surface $xyz=4$ at the point $(1,2,2)$?

(a) $x+2y+4z=12$

(b) $4x+2y+z=12$

(c) $x+4y+z=0$

(d) $2x+y+z=6$

Ans. : (d)

Solution: The surface equation is given by

$$\phi=xyz=4$$

The normal vector to the surface is

$$\vec{n}=\vec{\nabla}\phi=yz\hat{x}+xz\hat{y}+xy\hat{z}$$

At point $(1,2,2)$,

$$\vec{n}=(4\hat{x}+2\hat{y}+2\hat{z}),\ \hat{n}=\frac{\vec{n}}{|\vec{n}|}=\frac{(4\hat{x}+2\hat{y}+2\hat{z})}{\sqrt{16+4+4}}=\frac{(2\hat{x}+\hat{y}+\hat{z})}{\sqrt{6}}$$

The equation of plane at point $(1,2,2)$ is

$$\left[(x-1)\hat{x}+(y-2)\hat{y}+(z-2)\hat{z}\right]\hat{n}=0$$

$$\Rightarrow 2(x-1)+1(y-2)+1(z-2)=0\Rightarrow 2x+y+z=6$$

Q38. The integral $I=\int_1^{\infty}\frac{\sqrt{x-1}}{(1+x)^2}dx$ is

(a) $\frac{\pi}{\sqrt{2}}$

(b) $\frac{\pi}{2\sqrt{2}}$

(c) $\frac{\sqrt{\pi}}{2}$

(d) $\sqrt{\frac{\pi}{2}}$

Ans. : (b)

Solution: $I=\int_1^\infty \frac{\sqrt{x-1}}{(1+x)^2}dx$

Put, $x=(1+z^2),\ dx=2zdz$

Hence, $I=\int_0^\infty \frac{2z^2dz}{\left(2+z^2\right)^2}$

Here poles, $\left(2+z^2\right)=0\Rightarrow\left(z+i\sqrt{2}\right)\left(z-i\sqrt{2}\right)=0$

Only $\left(z=i\sqrt{2}\right)$ poles is allowed

Then $R\left(i\sqrt{2}\right)=\lim_{z\to i\sqrt{2}}\frac{1}{\sqrt{2-1}}\frac{d}{dz}\left[\frac{2z^2\left(z-i\sqrt{2}\right)^2}{\left(z-i\sqrt{2}\right)^2\left(z+i\sqrt{2}\right)^2}\right]$

$$=\lim_{z\to i\sqrt{2}}\left[\frac{\left(z+i\sqrt{2}\right)^2\cdot 4z-2z^2\cdot 2\left(z+i\sqrt{2}\right)}{\left(z+i\sqrt{2}\right)^4}\right]$$

$$=\frac{\left(2i\sqrt{2}\right)^2\times 4\left(i\sqrt{2}\right)-2\left(i\sqrt{2}\right)^2\cdot 2\left(2i\sqrt{2}\right)}{\left(2i\sqrt{2}\right)^4}=-\frac{32\sqrt{2}\,i+16\sqrt{2}\,i}{64}=-\frac{16\sqrt{2}\,i}{64}=-\frac{i}{2\sqrt{2}}$$

Hence, $\int_{-\infty}^{\infty}\frac{2z^2}{\left(2+z^2\right)^2}dz=2\pi i\left(-\frac{i}{2\sqrt{2}}\right)=\frac{\pi}{\sqrt{2}}$

$$\Rightarrow\int_0^\infty\frac{2z^2}{\left(2+z^2\right)^2}dz=\frac{\pi}{2\sqrt{2}}\Rightarrow\int_1^\infty\frac{\sqrt{x-1}}{(1+x)^2}dx=\frac{\pi}{2\sqrt{2}}$$

Q39. The Fourier transform of the function $\frac{1}{x^4+3x^2+2}$ up to proportionality constant is

(a) $\sqrt{2}\exp\left(-k^2\right)-\exp\left(-2k^2\right)$ (b) $\sqrt{2}\exp\left(-|k|\right)-\exp\left(-\sqrt{2}|k|\right)$

(c) $\sqrt{2}\exp\left(-\sqrt{|k|}\right)-\exp\left(-\sqrt{2|k|}\right)$ (d) $\sqrt{2}\exp\left(-\sqrt{2}k^2\right)-\exp\left(-2k^2\right)$

Ans. : (b)

Solution: $f(x)=\frac{1}{\left(x^4+3x^2+2\right)}=\frac{1}{\left(x^2+1\right)}-\frac{1}{\left[x^2+\left(\sqrt{2}\right)^2\right]}$

Now, Fourier transform of $f(x)$ is,

$$F(p)=A\int_{-\infty}^{\infty} f(x)e^{-1kx}dx$$

$$=A\int_{-\infty}^{\infty}\left[\frac{1}{\left(x^2+1\right)}-\frac{1}{x^2+\left(\sqrt{2}\right)^2}\right]e^{-ikx}dx=A\left[\int_{-\infty}^{\infty}\frac{1}{\left(x^2+1\right)}\times e^{-ikx}dx-\int_{-\infty}^{\infty}\frac{e^{-ikx}}{x^2+\left(\sqrt{2}\right)^2}dx\right]$$

$$\because \int_{-\infty}^{\infty}\frac{1}{\left(x^2+a^2\right)}e^{-ikx}dx=\sqrt{\frac{\pi}{2}}\frac{e^{-a|k|}}{a}$$

$$F(k)=A\left[\sqrt{\frac{\pi}{2}}\frac{e^{-|k|}}{1}-\sqrt{\frac{\pi}{2}}\frac{e^{-\sqrt{2}|k|}}{\sqrt{2}}\right]=\frac{A\sqrt{\pi}}{2}\left[\sqrt{2}\exp\left(-|k|\right)-\exp\left(-\sqrt{2}|k|\right)\right]$$

Q40. If $\rho=\frac{\left[I+\frac{1}{\sqrt{3}}\left(\sigma_x+\sigma_y+\sigma_z\right)\right]}{2}$, where σ's are the Pauli matrices and I is the identity matrix, then the trace of ρ^{2017} is

(a) 2^{2017} (b) 2^{-2017} (c) 1 (d) $\frac{1}{2}$

Ans. : (c)

Solution: Given, $\rho=\frac{\left[I+\frac{1}{\sqrt{3}}\left(\sigma_x+\sigma_y+\sigma_z\right)\right]}{2}$

Now, $\rho^2=\frac{1}{4}\left[I+\frac{1}{\sqrt{3}}\left(\sigma_x+\sigma_y+\sigma_z\right)\right]\left[I+\frac{1}{\sqrt{3}}\left(\sigma_x+\sigma_y+\sigma_z\right)\right]$

$$=\frac{1}{4}\left[I+\frac{2}{\sqrt{3}}\left(\sigma_x+\sigma_y+\sigma_z\right)+\frac{1}{3}\left(\sigma_x+\sigma_y+\sigma_z\right)^2\right]$$

$$=\frac{1}{4}\left[I+\frac{2}{\sqrt{3}}\left(\sigma_x+\sigma_y+\sigma_z\right)+\frac{1}{3}(3I)\right]$$

$$=\frac{1}{4}\left[2I+\frac{2}{\sqrt{3}}\left(\sigma_x+\sigma_y+\sigma_z\right)\right]=\frac{1}{2}\left[I+\frac{1}{\sqrt{3}}\left(\sigma_x+\sigma_y+\sigma_z\right)\right]$$

$\rho^2=\rho\Rightarrow\rho^n=\rho$., where n can be any positive integer

Therefore, $\rho^{2017}=\rho$

$$\rho^{2017}=\begin{bmatrix}\frac{1+1/\sqrt{3}}{2} & \frac{1-i}{2\sqrt{3}}\\ \frac{1+i}{2\sqrt{3}} & \frac{1-1/\sqrt{3}}{2}\end{bmatrix}$$

since, Trace of a matrix is equal to sum of their diagonal element, so

$$\text{Trace of } \rho^{2017}=\frac{1+\frac{1}{\sqrt{3}}}{2}+\frac{1-\frac{1}{\sqrt{3}}}{2}=1$$

Q41. The function $f(x)=\cosh x$ which exists in the range $-\pi\le x\le\pi$ is periodically repeated between $x=(2m-1)\pi$ and $(2m+1)\pi$, where $m=-\infty$ to ∞. Using Fourier series, indicate the correct relation at $x=0$

(a) $\sum\limits_{n=-\infty}^{\infty}\frac{(-1)^n}{1-n^2}=\frac{1}{2}\left(\frac{\pi}{\cosh\pi}-1\right)$

(b) $\sum\limits_{n=-\infty}^{\infty}\frac{(-1)^n}{1-n^2}=2\frac{\pi}{\cosh\pi}$

(c) $\sum\limits_{n=-\infty}^{\infty}\frac{(-1)^{-n}}{1+n^2}=2\frac{\pi}{\sinh\pi}$

(d) $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}=\frac{1}{2}\left(\frac{\pi}{\sinh\pi}-1\right)$

Ans. : (d)

Solution: $f(x)=\cosh x,\quad -\pi\le x\le\pi$

Here, $a_0=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\cosh x dx=\frac{1}{2\pi}[\sinh x]_{-\pi}^{\pi}=\frac{\sinh\pi}{\pi}$

$b_n=0$, due to even function

and $a_n=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\left(e^x+e^{-x}\right)\cos nx dx\qquad\left[\because \cosh x=\frac{1}{2}\left(e^x+e^{-x}\right)\right]$

$$a_n=\frac{1}{2\pi}\left[\frac{e^x}{(1+n^2)}(\cos nx+n\sin nx)+\frac{e^{-x}}{1+n^2}(-\cos nx+n\sin nx)\right]_{-\pi}^{\pi}$$

$$=\frac{1}{2\pi}\left[\frac{e^{\pi}(-1)^n}{(1+n^2)}-\frac{e^{-\pi}(-1)^n}{(1+n^2)}-\frac{e^{-\pi}(-1)^n}{(1+n^2)}+\frac{e^{\pi}(-1)^n}{(1+n^2)}\right]=\frac{2(-1)^n\cdot 2\sinh\pi}{2\pi(1+n^2)}=\frac{2(-1)^n\sinh\pi}{\pi(1+n^2)}$$

Hence, $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \Rightarrow \cosh x = \frac{\sinh \pi}{\pi} + \sum_{n=1}^{\infty} \frac{2(-1)^n \sinh \pi}{\pi(1+n^2)} \cos nx$

At $x = 0$,

$$\sum_{n=1}^{\infty} \frac{2(-1)^n \sinh \pi}{\pi(1+n^2)} = \left(1 - \frac{\sinh \pi}{\pi}\right) \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{(1+n^2)} = \frac{1}{2}\left[\frac{\pi}{\sinh \pi} - 1\right]$$

JEST-2018

Q42. For which of the following conditions does the integral $\int_0^1 P_m(x) P_n(x)\,dx$ vanish for $m \neq n$, where $P_m(x)$ and $P_n(x)$ are the Legendre polynomials of order m and n respectively?

(a) all $m, m \neq n$ (b) $m - n$ is an odd integer

(c) $m - n$ is a nonzero even integer (d) $n = m \pm 1$

Ans. : (c)

Solution: $\int_{-1}^{1} P_m(x) P_n(x)\,dx = \frac{2}{2x+1}\delta_{nm}$

$2\int_0^1 P_m(x) P_n(x)\,dx = \frac{2}{2x+1}\delta_{nm}$

Only $P_m(x) P_n(x) =$ even

m	
0	even
1	odd
2	even
3	odd
4	even

$m - n =$ non zero even integer then only

$P_m(x) P_n(x) =$ even

$m \neq n \; \Delta \delta_{nm} = 0$

Q43. The Laplace transform of $\frac{(\sin(at) - at\cos(at))}{(2a^3)}$ is

(a) $\frac{2as}{(s^2+a^2)^2}$ (b) $\frac{s^2-a^2}{(s^2+a^2)^2}$ (c) $\frac{1}{(s+a)^2}$ (d) $\frac{1}{(s^2+a^2)^2}$

Ans. : (d)

Solution: $L\left\{\frac{\sin at - at\cos at}{2a^3}\right\} = \frac{1}{(s^2+a^2)^2}$

Q44. Two of the eigenvalues of the matrix

$$A = \begin{pmatrix} a & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are 1 and -1. What is the third eigenvalue?

(a) 2 (b) 5 (c) -2 (d) -5

Ans. : (b)

Solution: The sum of eigenvalues $=$ Trace

Therefore $\lambda_1 + \lambda_2 + \lambda_3 = a + 2 + 1 = a + 3$

$\Rightarrow 1 - 1 + \lambda_3 = a + 3$

$\Rightarrow \lambda_3 = a + 3$ (i)

The product of eignevalues $=$ Determinant

$\Rightarrow \lambda_1 \cdot \lambda_2 \lambda_3 = 2a - 9$

$\Rightarrow (1)(-1)\lambda_3 = 2a - 9$

$\Rightarrow \lambda_3 = 9 - 2a$ (ii)

Eliminating a from equation (i) and putting it in (ii) gives

$\lambda_3 = 9 - 2(\lambda_3 - 3) \Rightarrow \lambda_3 = 9 - 2\lambda_3 + 6$

$\Rightarrow 3\lambda_3 = 15 \Rightarrow \lambda = 5$

Q45. $\pi\int_{-\infty}^{\infty}\exp(-|x|)\delta(\sin(\pi x))dx$, where $\delta(\ldots)$ is Dirac distribution, is

(a) 1 (b) $\frac{e+1}{e-1}$ (c) $\frac{e-1}{e+1}$ (d) $\frac{e}{e+1}$

Ans. : (b)

Solution: We know that $\delta[f(x)]=\sum_i\frac{\delta(x-x_i)}{|f'(x_i)|}$,

where x_i's are the roots of the equation $f(x)=0$.

Therefore, $\sin\pi x=0\Rightarrow\pi x=n\pi\Rightarrow x_i=n$ where n is an integer

$$f'(x)=\pi\cos\pi x\Rightarrow|f'(x_i)|=\left|\pi(-1)^n\right|=\pi$$

Hence, $\pi\int\limits_{-\infty}^{\infty}\exp(-|x|)\delta\sin(\pi x)dx$

$$=\int\limits_{-\infty}^{\infty}\exp(-|x|)\left[\delta(0)+\delta(x-1)+\delta(x-1)+\delta(x+1)+\delta(x-2)+\delta(x+2)+\cdots\right]$$

$$=e^{-|0|}+e^{-|1|}+e^{-|-1|}+e^{-|2|}+e^{-|-2|}+\cdots=1+e^{-1}+e^{-1}+e^{-2}+e^{-2}+\cdots$$

$$=1+2\left(e^{-1}+e^{-2}+e^{-3}+\cdots\right)$$

The terms in bracket form geometric series with first term e^{-1} and common ratio e^{-1}.

$$\pi\int\limits_{-\infty}^{\infty}\exp(-|-x|)\delta(\sin(\pi x))dx=1+2\cdot\frac{e^{-1}}{1-e^{-1}}=1+2\cdot\frac{1/e}{1-1/e}$$

$$=1+2\frac{1/e}{(e-1)/e}=1+2\cdot\frac{1}{e-1}=\frac{e-1+2}{e-1}=\frac{e+1}{e-1}$$

Q46. The integral

$$\int_{-\infty}^{\infty}\frac{\cos x}{x^2+1}dx \text{ is}$$

(a) $\frac{\pi}{e}$ (b) πe^{-2} (c) π (d) zero

Ans. : (a)

Solution: $f(z)=\frac{e^{iz}}{z^2+1}=\frac{e^{iz}}{(z+i)(z-i)}$

$$\int\limits_{-\infty}^{\infty}\frac{\cos x}{x^2+1}dx=\operatorname{Re}2\pi i\times\frac{e^{i\cdot i}}{zi}=\frac{\pi}{e}$$

Q47. An electronic circuit with 10000 components performs its intended function success fully with a probability 0.99 if there are no faulty components in the circuit. The probability that there are faulty components is 0.05. if there are faulty components, the circuit perform successfully with a probability 0.3. The probability that the circuit performs successfully is $\frac{x}{10000}$. What is x?

Ans. : 9555

Q48. If an abelian group is constructed with two distinct elements a and b such that $a^2 = b^2 = I$, where I is the group identity. What is the order order of the smallest abelian group containing a, b and I?

Ans. : 4

Solution: According to the question a, b and I are elements of group. The Cayley table for the group is

	I	a	b
I	I	a	b
a	a	I	ab
b	b	ba	I

For the commutative group $ab = ba$. If the order of the group is 3, then from the table we see that ab should be equal to b and ba should be equal to a.

$ab = b$ and $ba = a$

Using the commutative property we can write

$a = b$

But from the question a and b are distinct elements. Therefore the group will contain more than 3 elements.

The new Cayley-table is

	I	a	b	ab
I	I	a	b	ab
a	a	I	ab	b
b	b	ba	I	a
ab	ab	b	a	I

Here we have used the fact that for commutative group $ab = ba$.

From the table we see that this group is commutative and all group axioms are satisfied by the elements of group. So,

$$G = \{a, b, I, ab\}$$

The order of the group is 4.

Q49. If $F(x, y) = x^2 + y^2 + xy$, its Legendre transformed function $G(u, v)$, upto a multiplicative constant, is

(a) $u^2 + v^2 + uv$ (b) $u^2 + v^2 - uv$ (c) $u^2 + v^2$ (d) $(u+v)^2$

Ans. : (b)

Solution: $G = F - xu - yv$

$$dG = dF - xdu - udv - ydv - vdy$$

$$F = x^2 + y^2 + xy$$

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

$$dF = udx + vdy$$

$$dG = udx + vdy - xdu - udx - ydv - vdy$$

$$dG = -xdu - ydv$$

$$u = \frac{\partial F}{\partial x}, v = \frac{\partial F}{\partial x}$$

$$u = 2x + y, 2u = 4x + 2y$$

$$2v = 4y + 2x, v = 2y + x$$

$$y = +\frac{1}{3}[2v - u]$$

$$2u - v = 3x$$

$$x = \frac{1}{3}[2u - v]$$

$$dG = -\frac{1}{3}[2u - v]du - \frac{1}{3}[2v - u]dx$$

$$dG(u,v) = -\frac{1}{3}(2u-v)\,du - \frac{1}{3}(2v-u)\,dv = \frac{\partial G}{\partial u}du + \frac{\partial G}{\partial v}dv$$

$$\frac{\partial G}{\partial u} = -\frac{1}{3}(2u-v),\ \frac{\partial G}{\partial v} = -\frac{1}{3}(2v-u) \qquad \text{(i)}$$

$$G(u,v) = -\frac{1}{3}\left(u^2 - uv\right) + h(v) \qquad \text{(ii)}$$

$$G(u,v) = -\frac{1}{3}(-u) + \frac{dh(v)}{d(v)} = \frac{u}{3} + \frac{dh(v)}{dv}$$

$$\frac{-2v}{3} + \frac{u}{3} = \frac{u}{3} + \frac{dh(v)}{dx}$$

$$h(v) = \frac{-v^2}{3} \qquad \text{(iii)}$$

$$G(u,v) = -\frac{1}{3}\left(u^2 - uv\right) - \frac{v^2}{3} = -\frac{1}{3}\left(u^2 + v^2 - 4v\right)$$

JEST-2019

Q50. Let $\vec{r}$ be the position vector of a point on a closed contour C. What is the value of the line integral $\oint \vec{r}\cdot d\vec{r}$?

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) π

Ans. : (a)

Solution: $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \Rightarrow \vec{\nabla}\times\vec{r} = 0 \Rightarrow \oint \vec{r}\cdot d\vec{r} = 0$

Q51. Consider the function $f(x,y) = |x| - i|y|$. In which domain of the complex plane is this function analytic?

(a) First and second quadrants (b) Second and third quadrants

(c) Second and fourth quadrants (d) Nowhere

Ans. : (c)

Solution: $f(x,y) = |x| - i|y|$

$f(x,y) = x - iy = \overline{z}$

$f(x,y) = -x - iy = -z$

$f(x,y) = -x + iy = -\bar{z}$

$f(x,y) = x + iy = z$

We know $\bar{z}$ is not analytic and z and $-z$ are analytic. So answer is (c).

Q52. Suppose $\psi\vec{A}$ is a conservative vector, $\vec{A}$ is a non-conservative vector and ψ is non-zero scalar everywhere. Which one of the following is true?

(a) $(\nabla\times\vec{A})\cdot\vec{A} = 0$ (b) $\vec{A}\times\nabla\psi = \vec{0}$

(c) $\vec{A}\cdot\nabla\psi = 0$ (d) $(\nabla\times\vec{A})\times\vec{A} = \vec{0}$

Ans. : (a)

Solution: Divergence of a curl is always zero.

Q53. Consider two $n\times n$ matrices, A and B such that $A+B$ is invertible. Define two matrices, $C = A(A+B)^{-1}B$ and $D = B(A+B)^{-1}A$. Which of the following relations always hold true?

(a) $C = D$ (b) $C^{-1} = D$ (c) $BCA = ADB$ (d) $C \neq D$

Ans. : (a)

Solution: $C^{-1} = \left[A(A+B)^{-1}B\right]^{-1} = B^{-1}(A+B)A^{-1}$

$= B^{-1}AA^{-1} + B^{-1}BA^{-1} = B^{-1} + A^{-1}$

$\Rightarrow C^{-1} = B^{-1} + A^{-1}$

$D^{-1} = \left[B(A+B)^{-1}A\right]^{-1} = A^{-1}(A+B)B^{-1}$

$= A^{-1}AB^{-1} + A^{-1}B^{-1} = B^{-1} + A^{-1}$

or $D^{-1} = B^{-1} + A^{-1}$

From equation (i) and (ii)

$C^{-1} = D^{-1}$

or $CC^{-1}D = CD^{-1}D$ or $D = C$

Therefore, option (a) is correct.

Q54. Which one of the following vectors lie along the line of intersection of the two planes $x+3y-z=5$ and $2x-2y+4z=3$?

(a) $10\hat{i}-2\hat{j}+5\hat{k}$ (b) $10\hat{i}-6\hat{j}-8\hat{k}$

(c) $10\hat{i}+2\hat{j}+5\hat{k}$ (d) $10\hat{i}-2\hat{j}-5\hat{k}$

Ans. : (b)

Solution: Unit vector normal to $x+3y-z=5$ is $\hat{n}_1=\frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}=\frac{\hat{i}+3\hat{j}-\hat{k}}{\sqrt{1+9+1}}=\frac{\hat{i}+3\hat{j}-\hat{k}}{\sqrt{11}}$

Unit vector normal to $2x-2y+4z=3$ is $\hat{n}_2=\frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}=\frac{2\hat{i}-2\hat{j}+4\hat{k}}{\sqrt{4+4+16}}=\frac{2\hat{i}-2\hat{j}+4\hat{k}}{\sqrt{24}}$

Check for option (b) $\hat{n}=10\hat{i}-6\hat{j}-8\hat{k}$

$\hat{n}_1.\hat{n}=\frac{10-18+8}{\sqrt{11}}=0$ and $\hat{n}_2.\hat{n}=\frac{20+12-32}{\sqrt{24}}=0$

Q55. What is the value of the integral $\int_{-\infty}^{+\infty} dx\,\delta\left(x^2-\pi^2\right)\cos x$?

(a) π (b) $-\frac{1}{2\pi}$ (c) $-\frac{1}{\pi}$ (d) 0

Ans. : (c)

Solution: $\delta\left(x^2-\pi^2\right)=\frac{1}{|\pi-(-\pi)|}\left[\delta\left(x-\pi\right)+\delta\left(x+\pi\right)\right]$

$$=\frac{1}{2\pi}\left[\delta\left(x-\pi\right)+\delta\left(x+\pi\right)\right]$$

Therefore, $\int\limits_{-\infty}^{\infty} dx\,\delta\left(x^2-\pi^2\right)\cos x=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} dx\left[\delta\left(x-\pi\right)+\delta\left(x+\pi\right)\right]\cos x$

$$=\frac{1}{2\pi}\left[\cos\pi+\cos\left(-\pi\right)\right]=\frac{1}{2\pi}\left(-1-1\right)=-\frac{1}{\pi}$$

Q56. Let A be a hermitian matrix, and C and D be the unitary matrices. Which one of the following matrices is unitary?

(a) $C^{-1}AC$ (b) $C^{-1}DC$ (c) $C^{-1}AD$ (d) $A^{-1}CD$

Ans. : (b)

Solution: $\left(C^{-1}DC\right)\left(C^{-1}DC\right)^{\dagger} = C^{-1}DCC^{\dagger}D^{\dagger}\left(C^{-1}\right)^{\dagger}$

Since C is unitary $CC^{-1} = I$, therefore $\left(C^{-1}DC\right)\left(C^{-1}DC\right)^{\dagger} = C^{-1}DD^{\dagger}\left(C^{-1}\right)^{\dagger}$

Since D is unitary $DD^{\dagger} = I$, therefore, $\left(C^{-1}DC\right)\left(C^{-1}DC\right)^{\dagger} = C^{-1}\left(C^{-1}\right)^{\dagger}$

Since for any invertible matrix $\left(C^{-1}\right)^{\dagger} = \left(C^{+}\right)^{-1}$ we have

$\left(C^{-1}DC\right)\left(C^{-1}DC\right)^{\dagger} = C^{-1}\left(C^{\dagger}\right)^{-1}$

Since C is unitary $C^{\dagger} = C^{-1}$, therefore,

$\left(C^{-1}DC\right)\left(C^{-1}DC\right)^{\dagger} = C^{-1}\left(C^{-1}\right)^{-1} = C^{-1}C = I$

Therefore, $C^{-1}DC$ is a unitary matrix.

Q57. Consider a 2×2 matrix $A = \begin{pmatrix} 1 & 13 \\ 0 & 1 \end{pmatrix}$ what is A^{27}?

(a) $\begin{pmatrix} 1 & 13 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 13^{27} \\ 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 27 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 351 \\ 0 & 1 \end{pmatrix}$

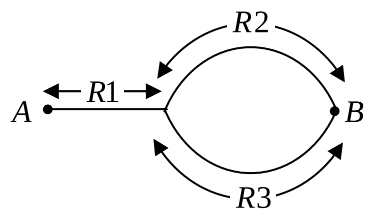
Ans. : (d)

Solution: Given $A = \begin{pmatrix} 1 & 13 \\ 0 & 1 \end{pmatrix}$, it can be easily proved (by mathematical induction) that

$$A^{n} = \begin{pmatrix} 1 & 13n \\ 0 & 1 \end{pmatrix}$$

For $n = 27$, $\quad A^{27} = \begin{pmatrix} 1 & 13.27 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 351 \\ 0 & 1 \end{pmatrix}$

Q58. A person plans to go from town A to town B by taking either the route $(R1+R2)$ with probability $\frac{1}{2}$ or the route $(R1+R3)$ with probability $\frac{1}{2}$ (see figure). Further, there is a probability $\frac{1}{3}$ that $R1$ is blocked, a probability $\frac{1}{3}$ that $R2$ is blocked, and a probability $\frac{1}{3}$ that $R3$ is blocked. What is the probability that he/she would reach town B?



(a) $\frac{8}{9}$ (b) $\frac{1}{3}$ (c) $\frac{4}{9}$ (d) $\frac{2}{3}$

Ans. : (c)

Solution: Given that probability of $R1$ blocked =1/3

Probability of $R1$ not blocked $= 1 - \frac{1}{3} = \frac{2}{3}$

Probability from A to B without restriction $= \frac{1}{2}$

Route $R2$ probability $= \frac{1}{2} \times \frac{2}{3}$ not blocked

Route $R3 = \frac{1}{2} \times \frac{2}{3}$

Total probability $(A \rightarrow B) = \frac{2}{3}\left[\frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times \frac{2}{3}\right] = \frac{4}{9}$

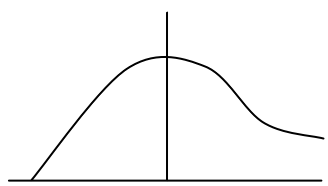
Q59. Consider a function $f(x) = P_k(x) e^{-(x^4+2x^2)}$ in the domain $x \in (-\infty, \infty)$, where P_k is any polynomial of degree k. What is the maximum possible number of extrema of the function?

(a) $k+3$ (b) $k-3$ (c) $k+2$ (d) $k+1$

Ans. : (a)

Solution: $f(x) = p_k(x) e^{-(x^4+2x^2)}$

Let $k = 0$, $f(x) = \rho_0(x) e^{-(x^4+2x^2)}$



Number of extrema

$P_0(x) = 1, k = 0$

Number of extrema $= 1$

$k + 1 = 0 + 1 = 1$

Q60. The Euler polynomials are defined by

$$\frac{2e^{xs}}{e^x + 1} = \sum_{n=0}^{\infty} E_n(s) \frac{x^n}{n!}$$

What is the value of $E_5(2) + E_5(3)$?

Ans. : 64

Solution: $\dfrac{2e^{xs}}{e^x+1}=\sum_{n=0}^{\infty}E_n(s)\dfrac{x^n}{n!}$

$$E_n(x+1)+E_n(x)=2x^n$$

$$E_5(x+1)+E_5(x)=2x^5$$

$$x=2=2\times 2^5=64$$

Q61. What is the angle (in degrees) between the surfaces $y^2+z^2=2$ and $y^2-x^2=0$ at the point $(1,-1,1)$

Ans. : 60

Solution: The equations of two surfaces are

$$f(x,y,z)=2 \text{ and } g(x,y,z)=0$$

where $f(x.y,z)=y^2+z^2$ and $g(x,y,z)=y^2=x^2$

The normal to the first surfaces is

$$\overrightarrow{\nabla f}=\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}\Rightarrow\overrightarrow{\nabla f}=2y\hat{j}+2z\hat{k}$$

$$\overrightarrow{\nabla g}=\frac{\partial g}{\partial x}\hat{i}+\frac{\partial g}{\partial y}+\hat{j}+\frac{\partial g}{\partial z}\hat{k}\Rightarrow\overrightarrow{\nabla g}=-2x\hat{i}+2y\hat{j}$$

At point $(1,-1,1)$, $\overrightarrow{\nabla f}=-2\hat{j}+2\hat{k}$ and $\overrightarrow{\nabla g}=-2\hat{i}-\hat{j}$

Hence the angle between the two surfaces is

$$\theta=\cos^{-1}\frac{\overrightarrow{\nabla f}\cdot\overrightarrow{\nabla g}}{\left|\overrightarrow{\nabla f}\right|\left|\overrightarrow{\nabla g}\right|}=\cos^{-1}\frac{\left(-2\hat{j}+2\hat{k}\right)\cdot\left(-2\hat{i}-2\hat{j}\right)}{\sqrt{8}\sqrt{8}}$$

or $\qquad\theta=\cos^{-1}\dfrac{4}{8}=\cos^{-1/2}=60^0$